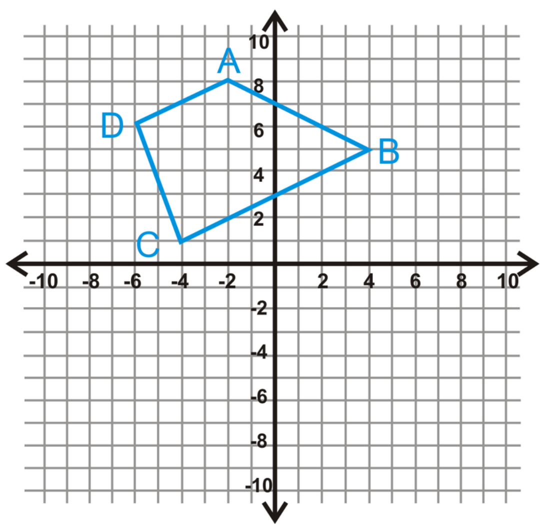
9.5 Composition of Transformations

Learning Objectives

- Perform a glide reflection.
- Perform a reflection over parallel lines and the axes.
- Perform a double rotation with the same center of rotation.
- Determine a single transformation that is equivalent to a composite of two transformations.

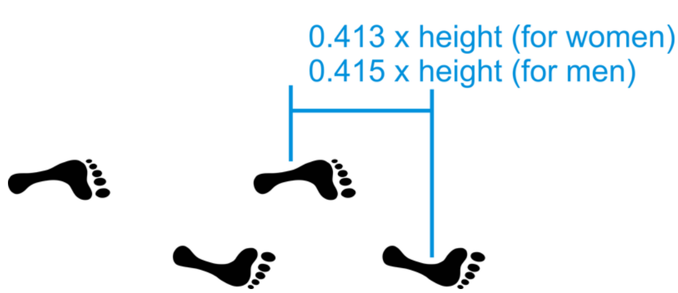
Review Queue

a. Reflect $ABCD$ over the $x-$axis. Find the coordinates of $A'B'C'D'$.



b. Translate $A'B'C'D'$ such that $(x,y) \rightarrow (x+4,y)$. Find the coordinates of $A''B''C''D''$.
c. Now, start over. Translate $ABCD$ such that $(x,y) \rightarrow (x+4,y)$. Find the coordinates of $A'B'C'D'$.
d. Reflect $A'B'C'D'$ from #3 over the $x-$axis. Find the coordinates of $A''B''C''D''$. Are they the same as #2?

Know What? An example of a glide reflection is your own footprint. The equations to find your average footprint are in the diagram below. Determine your average footprint and write the rule for one stride. You may assume your stride starts at (0, 0).



Glide Reflections

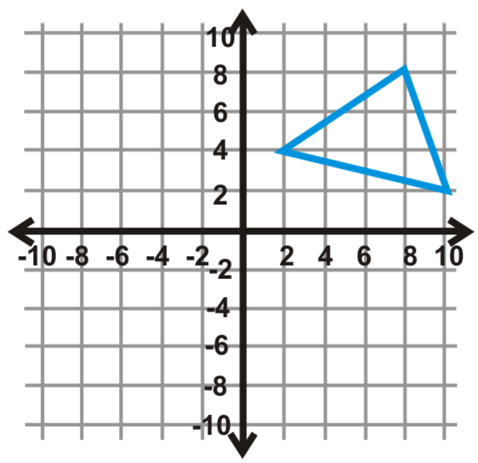
Now that we have learned all our rigid transformations, or isometries, we can perform more than one on the same figure. In your homework last night you actually performed a composition of two reflections. And, in the Review Queue above, you performed a composition of a reflection and a translation.

Composition (of transformations): To perform more than one rigid transformation on a figure.

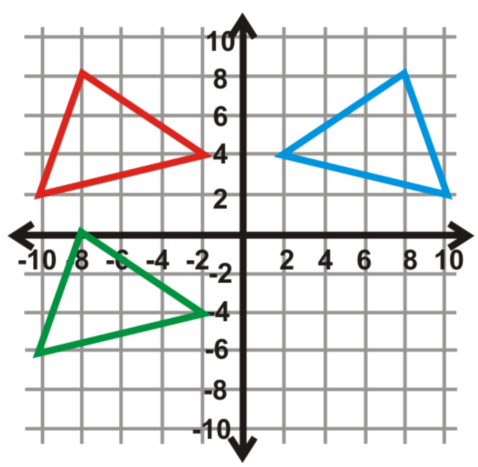
Glide Reflection: A composition of a reflection and a translation. The translation is in a direction parallel to the line of reflection.

So, in the Review Queue above, you performed a glide reflection on $ABCD$. Hopefully, in #4, you noticed that ***the order in which you reflect or translate does not matter.*** It is important to note that the translation for any glide reflection will always be in one direction. So, if you reflect over a vertical line, the translation can be up or down, and if you reflect over a horizontal line, the translation will be to the left or right.

Example 1: Reflect $\triangle ABC$ over the $y-$axis and then translate the image 8 units down.



Solution: The green image below is the final answer.



$$A(8,8) \rightarrow A''(-8,0)$$
$$B(2,4) \rightarrow B''(-2,-4)$$
$$C(10,2) \rightarrow C''(-10,-6)$$

One of the interesting things about compositions is that they can always be written as one rule. What this means is, you don't necessarily have to perform one transformation followed by the next. You can write a rule and perform them at the same time.

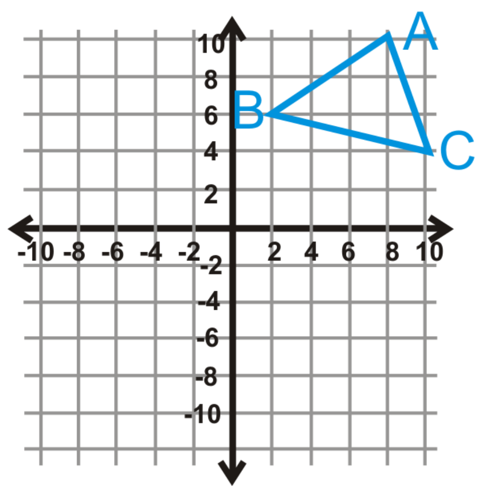
Example 2: Write a single rule for $\triangle ABC$ to $\triangle A''B''C''$ from Example 1.

Solution: Looking at the coordinates of A to A'', the $x-$value is the opposite sign and the $y-$value is $y-8$. Therefore the rule would be $(x,y) \rightarrow (-x,y-8)$.

Notice that this follows the rules we have learned in previous sections about a reflection over the $y-$axis and translations.

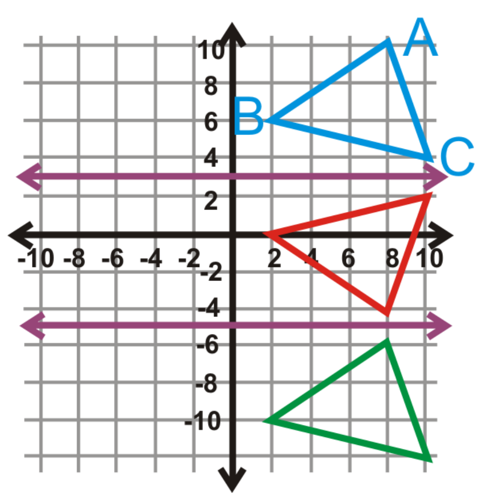
Reflections over Parallel Lines

The next composition we will discuss is a double reflection over parallel lines. For this composition, we will only use horizontal or vertical lines.



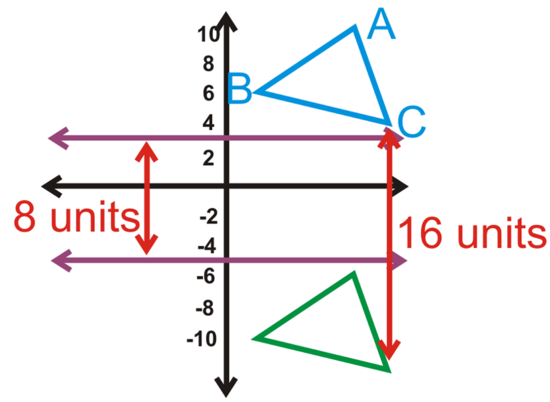
Example 3: Reflect $\triangle ABC$ over $y = 3$ and $y = -5$.

Solution: Unlike a glide reflection, order matters. Therefore, you would reflect over $y = 3$ first, followed by a reflection of this image (red triangle) over $y = -5$. Your answer would be the green triangle in the graph below.



Example 4: Write a single rule for $\triangle ABC$ to $\triangle A''B''C''$ from Example 3.

Solution: Looking at the graph below, we see that the two lines are 8 units apart and the figures are 16 units apart. Therefore, the double reflection is the same as a single translation that is double the distance between the two lines.



$$(x,y) \rightarrow (x,y-16)$$

Reflections over Parallel Lines Theorem: If you compose two reflections over parallel lines that are h units apart, it is the same as a single translation of $2h$ units.

Be careful with this theorem. Notice, it does not say which direction the translation is in. So, to apply this theorem, you would still need to visualize, or even do, the reflections to see in which direction the translation would be.

Example 5: $\triangle DEF$ has vertices $D(3,-1), E(8,-3)$, and $F(6,4)$. Reflect $\triangle DEF$ over $x=-5$ and $x=1$. This double reflection would be the same as which one translation?

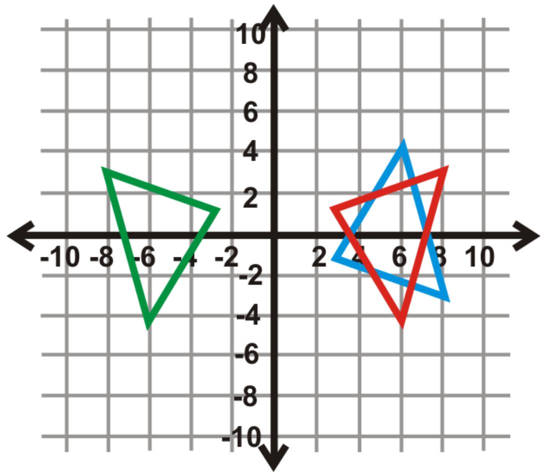
Solution: From the Reflections over Parallel Lines Theorem, we know that this double reflection is going to be the same as a single translation of $2(1-(-5))$ or 12 units. Now, we need to determine if it is to the right or to the left. Because we first reflect over a line that is further away from $\triangle DEF$, to the ***left***, $\triangle D''E''F''$ will be on the ***right*** of $\triangle DEF$. So, it would be the same as a translation of 12 units to the right. If the lines of reflection were switched and we reflected the triangle over $x=1$ followed by $x=-5$, then it would have been the same as a translation of 12 units to the ***left.***

Reflections over the x and y Axes

You can also reflect over intersecting lines. First, we will reflect over the x and y axes.

Example 6: Reflect $\triangle DEF$ from Example 5 over the $x-$axis, followed by the $y-$axis. Determine the coordinates of $\triangle D''E''F''$ and what one transformation this double reflection would be the same as.

Solution: $\triangle D''E''F''$ is the green triangle in the graph below. If we compare the coordinates of it to $\triangle DEF$, we have:



$$D(3,-1) \rightarrow D'(-3,1)$$
$$E(8,-3) \rightarrow E'(-8,3)$$
$$F(6,4) \rightarrow F'(-6,-4)$$

If you recall the rules of rotations from the previous section, this is the same as a rotation of 180°.

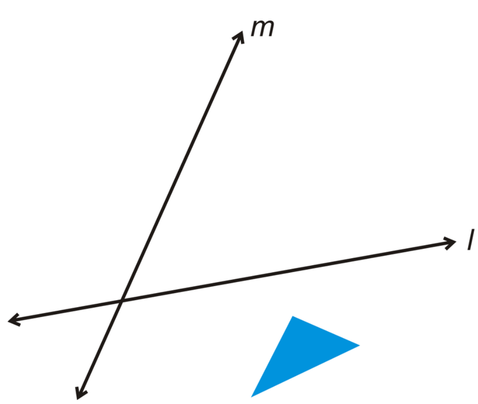
Reflection over the Axes Theorem: If you compose two reflections over each axis, then the final image is a rotation of 180° of the original.

With this particular composition, order does not matter. Let's look at the angle of intersection for these lines. We know that the axes are perpendicular, which means they intersect at a 90° angle. The final answer was a rotation of 180°, which is double 90°. Therefore, we could say that the composition of the reflections over each axis is a rotation of double their angle of intersection.

Reflections over Intersecting Lines

Now, we will take the concept we were just discussing and apply it to any pair of intersecting lines. For this composition, we are going to take it out of the coordinate plane. Then, we will apply the idea to a few lines in the coordinate plane, where the point of intersection will always be the origin.

Example 7: Copy the figure below and reflect it over l, followed by m.



Solution: The easiest way to reflect the triangle is to fold your paper on each line of reflection and draw the image. It should look like this: